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LETTER TO THE EDITOR

The Painlevé property transformed

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Abstract. Noteworthy analytic properties of the Dym–Kruskal equation, such as the weak Painlevé property and the Pogrebkov property (a second-order pole at infinity), are proven to be exactly the image of the usual Painlevé property under the Ibragimov transformation. The transformed Painlevé analysis is carried out for two infinite sets of evolution equations similar to the Dym–Kruskal equation. Only four of the equations pass the test, being known and integrable.

Despite the absence of any proof, the Painlevé property (PP) for partial differential equations (PDEs) [1] is generally believed to be a sufficient condition for 'integrability' of PDEs [2]. Almost all PDEs known to possess the PP are either integrable by some modifications of the inverse scattering transform or exactly linearizable (see e.g. [2, 3] and references therein). Probably the most impressive confirmation of a sufficiency of the PP for integrability of PDEs was given in [4] where the Painlevé analysis of a three-parameter class of evolution equations selected integrable cases only.

When the PP is used for detecting integrable PDEs, the analysis of all possible movable singularities is of crucial importance [2]. Many authors tested only those points at which a dependent variable went to infinity but ignored points at which the order or the type of a PDE changed (e.g. appendix A in [1] and [5]); certainly, this caused criticism in obviously non-integrable cases only [6]. Moreover, since the Painlevé analysis of PDEs is usually realized by means of power expansions, one needs much caution in order not to overlook essential singularities. For instance, both equations $u_t = u_{xxx}$ and $u_t = u_{xxx} + u_x^3$ pass the Painlevé test [1] 'by default', but the former, being linear, has no movable singularities, whereas the latter has a movable dominant logarithmic singularity in its general solution. In this letter, we use the Weiss–Kruskal algorithm (as formulated in [2]) which is sensitive to algebraic and non-dominant logarithmic branch points and simple in calculations.

The Dym–Kruskal equation [7]

$$v_t = v^3 v_{yyy} \tag{1}$$

$v = v(y, t)$, has no PP [8]. However, (1) has other noteworthy analytic properties such as the weak PP [8, 2] and the Pogrebkov property [9]. The weak PP means that algebraic branch points are the only movable singularities of the general solution of (1), i.e. Weiss–Kruskal expansion $v = \sum_{k=0}^{\infty} \delta_k(t)(y - \beta(t))^{(2+k)/3}$ satisfies (1) with arbitrary β, δ_2 and δ_4. The Pogrebkov property means that the general solution of (1) has a second-order pole at the infinitely removed point, i.e. expansion $v = \sum_{k=0}^{\infty} \delta_k(t) y^{2-k}$ satisfies (1) with arbitrary δ_0, δ_1 and δ_2. These properties differ from the usual PP

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which bans movable branch points and fixes no structure of a solution at infinity. What is the nature of such exotic analytic properties of the Dym–Kruskal equation? In this letter, they are proven to be exactly the image of the usual PP under a certain map of evolution equations. After proper definition, such a 'transformed Painlevé property' (TPP) is applied to analysis of two infinite sets of evolution equations.

The Ibragimov transformation [10]

$$(x, t, u(x, t)) \to (y, t, v(y, t)) \qquad y = u \qquad v = u_x \tag{2}$$

maps the equation

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2 \tag{3}$$

to the Dym–Kruskal equation (1). The polynomial form of (3) $u_x u_{xxx} - \frac{3}{2} u_{xx}^2 - u_x u_t = 0$ indicates that the Painlevé test for (3) consists in analysis of two possible singularities, $u \to \infty$ and $u_x \to 0$. The usual way [2] of determining leading behaviours of solutions, constructing recursion relations, finding resonances and checking compatibility conditions at resonances brings us to the corresponding Weiss–Kruskal expansions:

$$u = \varepsilon_0 (x-\alpha)^{-1} + \varepsilon_1 + \tfrac{1}{6}\varepsilon_0 \alpha' (x-\alpha) + \tfrac{1}{24}\varepsilon_0' (x-\alpha)^2 + \ldots \tag{4}$$

$$u = \varepsilon_0 - \tfrac{1}{12}\varepsilon_0' (x-\alpha)^3 - \tfrac{1}{120}\varepsilon_0' \alpha' (x-\alpha)^5 + \tfrac{1}{288}\varepsilon_0'' (x-\alpha)^6 + \varepsilon_7 (x-\alpha)^7 + \ldots \tag{5}$$

where the prime denotes the t-derivative, functions $\alpha(t)$, $\varepsilon_0(t)$ and $\varepsilon_1(t)$ are arbitrary in (4), and functions $\alpha(t)$, $\varepsilon_0(t)$ and $\varepsilon_7(t)$ are arbitrary in (5). Thus, (3) has the PP. Let us apply transformation (2) to expansion (4). Since $y = u$, (4) determines y as a function of x and t, $y(x, t)$. Differentiating (4) with respect to x, we get

$$v(x, t) = u_x = -\varepsilon_0 (x-\alpha)^{-2} + \tfrac{1}{6}\varepsilon_0 \alpha' + \tfrac{1}{12}\varepsilon_0' (x-\alpha) + \ldots .$$

Removing step by step x from expansions $y(x, t)$ and $v(x, t)$, we find $v(y, t)$:

$$v = -\varepsilon_0^{-1} y^2 + 2\varepsilon_0^{-1}\varepsilon_1 y + (\tfrac{1}{2}\varepsilon_0 \alpha' - \varepsilon_0^{-1}\varepsilon_1^2) + \tfrac{1}{6}\varepsilon_0 \varepsilon_0' y^{-1} + \ldots . \tag{6}$$

The same manipulations with (5) bring us to

$$v = -(\tfrac{3}{2})^{2/3} (\varepsilon_0')^{1/3} (y-\varepsilon_0)^{2/3} - \tfrac{2}{5}(\tfrac{3}{2})^{4/3} (\varepsilon_0')^{-1/3} \alpha' (y-\varepsilon_0)^{4/3} - (\tfrac{2}{3})^{1/3} (\varepsilon_0')^{-5/3} \varepsilon_0'' (y-\varepsilon_0)^{5/3}$$
$$+ (720 (\varepsilon_0')^{-2} \varepsilon_7 + \tfrac{11}{25} (\alpha')^2 (\varepsilon_0')^{-1}) (y-\varepsilon_0)^2 + \ldots . \tag{7}$$

Coefficients at y^2, y^1 and y^0 in (6) are arbitrary due to arbitrariness of ε_0, ε_1 and α'. In (7), coefficients at $(y-\varepsilon_0)^{4/3}$ and $(y-\varepsilon_0)^2$ are arbitrary due to arbitrariness of α' and ε_7; also function ε_0 is arbitrary which determines the 'singularity manifold'. Expansions (6) and (7) obviously express the Pogrebkov property and the weak PP of the Dym–Kruskal equation respectively, and these properties are exactly the image of the usual PP under the Ibragimov transformation. Vice versa, the PP of (3) can be derived from the expansions expressing the analytical properties of (1). This is achieved by the inversion of (2) $x = \int (v(u, t))^{-1}\, du + \alpha(t)$ (α is a 'constant' of integration) which determines an expansion of u by powers of $x - \alpha(t)$.

Let us generalize our observations. Suppose an expansion like (4) describes a pole of order p, i.e. $u = \varepsilon_0(t)(x-\alpha(t))^{-p} + \ldots$, $p = 1, 2, 3, \ldots$. Then the corresponding expansion like (6) $v = \delta_0(t) y^{(p+1)/p}$ expresses the Pogrebkov property (a second-order pole at infinity) at $p = 1$ only, whereas at $p > 1$ it describes an infinitely removed algebraic branch point of degree $-(p+1)/p$. Any other branching of v at infinity leads to movable branch points of u banned by the PP. Similarly, expansion $u = \varepsilon_0(t) + \varepsilon_{p+1}(t)(x-\alpha(t))^{p+1} + \ldots$, $p = 1, 2, 3, \ldots$, which describes the behaviour of u at

$u_x \to 0$, corresponds to $v = \delta_0(t)(y-\beta(t))^{p/(p+1)} + \ldots$. Any other movable branching of v leads to banned branching of u. This generalization is necessary because the Ibragimov transformation connects very wide classes of evolution equations. One can prove easily that (2) maps any equation

$$u_t = x\sigma(t)u_1 + u_1 s(t, u, u_1, u_1^{-1}u_2, \ldots, (u_1^{-1}D_x)^{N-1}u_1) \tag{8}$$

to corresponding equation

$$v_t = \sigma(t)v + v^2 D_y s(t, y, v, v_1, \ldots, v_{N-1}) \tag{9}$$

where order N and functions σ and s are arbitrary, $u_k = \partial^k u/\partial x^k$, $v_k = \partial^k v/\partial y^k$, $k = 0, 1, 2, \ldots$, $u_0 = u$, $v_0 = v$, $D_x = \partial/\partial x + \Sigma_{k=0}^{\infty} u_{k+1}\partial/\partial u_k$ and $D_y = \partial/\partial y + \Sigma_{k=0}^{\infty} v_{k+1}\partial/\partial v_k$. Let us confine ourselves to such equations (8) for which the Painlevé test consists in analysis of singularities at $u \to \infty$ and $u_x \to 0$ only. (This is achieved, e.g., if right-hand sides of corresponding equations (9) are differential polynomials of form $v^M v_N + \ldots$.) Then the same consideration as for equations (1) and (3) allows us to prove that (8) *has the* PP *if and only if (9) has the* TPP formulated as follows:

(i) Every q (Re $q > 1$), such that $v = \delta_0(t)y^q + \ldots$ is the leading behaviour of a solution of (9) at $y \to \infty$, satisfies $q = (p+1)/p$ with positive integer p, and expansion $v = \Sigma_{k=0}^{\infty} \delta_k(t) y^{(p+1-k)/p}$ satisfies (9) with compatible recursion relations. For at least one such q, the expansion contains N arbitrary functions.

(ii) Every q ($0 < \text{Re}\, q < 1$), such that $v = \delta_0(t)(y-\beta(t))^q + \ldots$ is the leading behaviour of a solution of (9) at $y - \beta(t) \to 0$, satisfies $q = p/(p+1)$ with positive integer p, and expansion $v = \Sigma_{k=0}^{\infty} \delta_k(t)(y-\beta(t))^{(p+k)/(p+1)}$ satisfies (9) with compatible recursion relations. For at least one such q, the expansion contains N arbitrary functions.

Let us consider the following evolution equation, similar to the Dym–Kruskal one,

$$w_t = w^{\mu} w_N \tag{10}$$

where $w = w(y, t)$, $w_N = \partial^N w/\partial y^N$, constant μ is arbitrary, $N > 1$. Let (10) have an infinite algebra of generalized symmetries [11], as the Dym–Kruskal equation does [12]. Then, according to [13], equation (10) has conserved density $w^{-\mu/N}$ necessarily, i.e. a function $f(w, w_1, \ldots, w_{N-1})$ exists such that $(w^{-\mu/N})_t = D_y f$. Since the kernel of the Euler operator $E = \Sigma_{k=0}^{\infty}(-1)^k D_y^k \partial/\partial w_k$ coincides with the image of D_y [11], we have the equivalent condition $E(w^{\nu} w_N) \equiv 0$, where $\nu = \mu - \mu/N - 1$, which is satisfied if and only if $\nu = 0$ at $N = 2, 3, 4, 5, \ldots$ and $\nu = 1$ at $N = 3, 5, 7, 9, \ldots$. The selected equations can be represented in polynomial form $v_t = v^N v_N + \ldots$ by substitution $v = w^{\mu/N}$:

$$v_t = n^{-1} v^2 D_y^{n+1} v^n \tag{11}$$

$$v_t = n^{-1} v^{n+2} D_y^{2n+1} v^n \tag{12}$$

$n = 1, 2, 3, 4, \ldots$. Equations (11) at $n = 1, 2$ and (12) at $n = 1, 2$ do possess infinite algebras of generalized symmetries [12–14]. However, one knows nothing about the equations at $n > 2$. Taking into account that (12) at $n = 1$ is the Dym–Kruskal equation, let us investigate which of equations (11) and (12) possess the TPP too. Equations (11) and (12) belong to class (9) at $\sigma = 0$. However, the corresponding equations of class (8) are very complicated at large n. Therefore the transformed Painlevé analysis of (11) and (12) is easier than the equivalent usual Painlevé analysis of corresponding equations $u_t = u_N + \ldots$.

Equations (11) satisfy condition (i) of the TPP 'by default', i.e. their solutions have no leading behaviour $v = \delta_0(t)y^q + \ldots$ with Re $q > 1$. This is a dangerous symptom indicating that corresponding equations $u_t = u_{n+1} + \ldots$ may possess movable dominant essential singularities at $u \to \infty$. However, condition (ii) solves the problem. Indeed, admissible q are $1/n, 2/n, \ldots, (n-1)/n$ and $n/(n+1)$ there, and some of them have no required form $p/(p+1)$ when $n>2$. This means that corresponding equations $u_t = u_{n+1} + \ldots$ at $n>2$ admit movable algebraic branch points at $u_x \to 0$. The remaining equations, i.e. (11) for $n=1$ and 2, $v_t = v^2 v_2$ and $v_t = v^3 v_3 + 3v^2 v_1 v_2$, correspond to linear ones, $u_t = u_2$ and $u_t = u_3$, and therefore need no further analysis.

Equations (12) also possess the TPP only if $n=1$ and 2. Indeed, condition (i) of the TPP gives $q=(n+1)/n, (n+2)/n, \ldots, 2$ which leads to movable algebraic branch points for corresponding equations $u_t = u_{2n+1} + \ldots$ when $n>2$. The Dym–Kruskal equation ($n=1$) has the TPP, and we need to investigate (12) at $n=2$ only:

$$v_t = v^5 v_5 + 5v^4(v_1 v_4 + 2v_2 v_3). \tag{13}$$

Checking condition (i) of the TPP for (13), we find that $q=\frac{3}{2}$ and 2, $p=2$ and 1, recursion relations are compatible and determine expansions $v=\sum_{k=0}^{\infty} \delta_k(t) y^{(3-k)/2}$ with four arbitrary functions δ_0, δ_2, δ_4, δ_6 and $v=\sum_{k=0}^{\infty}\delta_k(t)y^{2-k}$ with five arbitrary functions δ_0, δ_1, δ_2, δ_3, δ_4. Condition (ii) is also satisfied for (13), namely: $q=\frac{4}{5}$ and $\frac{1}{2}$, $p=4$ and 1, recursion relations are compatible and determine expansions $v=\sum_{k=0}^{\infty}\delta_k(t)(y-\beta(t))^{(k+4)/5}$ with four arbitrary functions β, δ_4, δ_6, δ_{12} and $v=\sum_{k=0}^{\infty}\delta_k(t)(y-\beta(t))^{(k+1)/2}$ with five arbitrary functions β, δ_0, δ_2, δ_4, δ_6. Thus, (13) has the TPP.

Equations (11) and (12) possess the TPP only if $n=1$ or 2. Does it mean that the equations are non-integrable at $n>2$? Certainly, no. Any analytic property of PDEs is strongly non-invariant under transformations of PDEs and cannot be considered as a necessary condition for integrability therefore. On the contrary, many exotic analytic properties of PDEs can be sufficient for integrability to the same extent as the PP. The correspondence between the PP and the TPP demonstrates this.

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